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ON THE PERIPHERAL SPECTRUM OF POSITIVE OPERATORS

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ABSTRACT

We prove that if E is a Banach lattice and $S, T \in \mathscr{L}(E)$ are such that $0 \leqq s \leqq T$, $r(s) = r(T)$ and $r(T)$ is a Riesz point of $\sigma(T)$ then $r(S)$ is a Riesz point of $\sigma(S)$. We prove also some results on compact positive perturbations of positive irreducible operators and lattice homomorphisms.

1. Introduction

An interesting problem in the theory of positive operators in Banach lattices is to know what properties of $T \in \mathscr{L}(E)$, where E is a Banach lattice, are inherited by $S \in \mathscr{L}(E)$ if we know that $0 \leqq S \leqq T$. Topological properties have been considered, for example, in [1], [5] or [16] (see [23], chapter 18 for a survey). The spectral properties are also interesting. We consider in this paper the following problem: Let E be a Banach lattice. Let $S, T \in \mathscr{L}(E)$ be such that $0 \leqq S \leqq T$ and $r(S) = r(T)$. If $r(T)$ is a Riesz point of $\sigma(T)$, i.e. if $r(T)$ is a pole of the resolvent $(z - T)^{-1}$ whose residuum is a projection of finite rank, is $r(S)$ a Riesz point of $\sigma(S)$? The answer is affirmative (Theorem 4.1 below). Two main facts are involved in the proof. One of them is the proof of theorem V-5.5 in [19] ("If E is a Banach lattice and $r(T)$ is a Riesz point of $\sigma(T)$, $0 \leqq T \in \mathscr{L}(E)$, then the set $(z \in \sigma(T): |z| = r(T))$ is a set of poles of $(z - T)^{-1}$"). The second fact is Theorem 4.3 below based on some considerations on spectral theory to be discussed in Section 2. To be more precise, let us fix a free ultrafilter $\mathscr{U}$ on $\mathbf{N}$ containing the Frechet filter. If E is a Banach space and $T \in \mathscr{L}(E)$, we denote by $\hat{E}$ the ultrapower of E with respect to the ultrafilter $\mathscr{U}$ and let us denote by $\hat{T}$ the canonical extension of T as an operator on $\hat{E}$ (see [19] for a short description). Then, in Section 2, we define, for a dual Banach space E, a

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projection $m_E: \hat{E} \to E$ by $\langle m_E(\hat{u}), g\rangle = \lim_{\mathscr{U}} \langle u_n, g\rangle$, $\hat{u} = (u_n)_{\mathscr{U}} \in \hat{E}$, $g \in E_* :=$ a predual of E, that will play a role in the proof of Theorem 4.3. This projection turns out to be useful for some other purposes. Indeed, in Section 2, we prove the cyclicity of the peripheral essential spectrum of a positive dual operator, satisfying some additional standard conditions, acting on a dual Banach lattice of measurable functions defined on a discrete measure space (Corollary 2.7). Weis and Wolff proved a similar result for operators defined on $L^1([0,1])$ ([20]). In Section 3, the projection m_E will be used to obtain a characterization of $\partial_\infty\sigma(T) \cap \sigma_{ess}(T)$ (Theorem 3.4 below) where $\partial_\infty\sigma(T)$ denotes the set of points $z \in \sigma(T)$ such that all neighborhoods of z intersect the unbounded connected component of the resolvent set $\rho(T)$ and $\sigma_{ess}(T)$ denotes the essential spectrum of T and T is a dual operator defined on a dual Banach space. Precisely, in that condition, for $z \in \partial_\infty\sigma(T)$, $z \in \sigma_{ess}(T)$ iff there exists $\hat{y} \in \hat{E}$, $\hat{y} \neq 0$, such that $\hat{T}\hat{y} = z\hat{y}$ and $m_E(\hat{y}) = 0$. We prove also in Section 3 for $z \in \partial_\infty\sigma(T)$ that $z \in \sigma_{ess}(T)$ iff $\dim \operatorname{Ker}(z - \hat{T})$ is finite basing it on a previous result of Groh ([7], Prop. 3.2). This is a refinement of a result in ([3]) and was proved by Wolff in ([22]) for representations of groups in Banach lattices using non-standard analysis. We finish our paper (Section 5) by studying the compact positive perturbations of certain positive operators (irreducible, lattice homomorphisms) and a similar problem for the generator of a strongly continuous semigroup of positive operators defined on a Banach lattice.

2. A projection from E onto E and some consequences

In the sequel by Banach space we mean a complex Banach space. We say that the decomposition of the Banach space F, $F = F_1 + F_2$, reduces the operator $T \in \mathscr{L}(F)$ if both spaces F_1 and F_2 are invariant under T. From now on we fix a free ultrafilter $\mathscr{U}$ on $\mathbf{N}$ containing the Frechet filter. If E is a Banach space, we denote by $\hat{E}$ the ultrapower of E with respect to the ultrafilter $\mathscr{U}$. If $T \in \mathscr{L}(E)$, $\hat{T}$ will be the canonical extension of T as an operator on $\hat{E}$.

PROPOSITION 2.1. *Let E be a dual Banach space. Then, E is isomorphic to a complemented subspace of $\hat{E}$ by a projection m_E of norm one. If E is a dual Banach lattice, we can take m_E to be positive. Moreover, if $T \in \mathscr{L}(E)$ is a dual operator, the decomposition of E, $\hat{E} = E + \operatorname{Ker} m_E$ reduces T.*

PROOF. Let E_* be a predual of E. Let $\hat{x} \in \hat{E}$. Choose a sequence (x_n) in E representing the vector $\hat{x} \in \hat{E}$. Let $g \in E_*$. The map $g \to \lim_{\mathscr{U}} \langle x_n, g\rangle$ from E_* into $\mathbf{C}$ is a continuous linear functional on E_*, so that there exists an element

$m_E(x) \in E$ such that $\langle m_E(x), g\rangle = \lim_{\mathscr{U}} \langle x_n, g\rangle$. It is straightforward to prove that the mapping from $\hat{E}$ onto E given by $\hat{x} \to m_E(\hat{x})$ is a well defined bounded linear projection of norm one from $\hat{E}$ onto E extending the identity operator on $\hat{E}$. The other assertions are immediate and we shall omit its proof.

Before stating the next proposition, let us make the terminology precise. We say that E is a sequence Banach lattice if E has a Schauder basis (e_n) such that $x = \sum_{n=1}^{\infty} x_n e_n \geqq 0$ iff $x_n \geqq 0$ for all $n \in \mathbf{N}$. For each $m \in \mathbf{N}$, define $S_m x = \sum_{n=1}^{m} x_n e_n$ and $Q_m x = x - S_m x$. The second assertion in Proposition 2.2 will not be used below but we include it because we find it interesting.

PROPOSITION 2.2. *Let I be an index set. Let E be a Banach lattice of functions $x: I \to \mathbf{C}$. Assume that E is a dual Banach space and the evaluations $e_i^*(x) := x(i)$ determine elements of a predual of E: E_*. Then,* $\operatorname{Ker} m_E$ *is a sublattice of E. If E is a sequence Banach lattice with predual E_* whose functionals e_n^* associated to the Schauder basis of E, (e_n), $e_m^*(\sum_{n=1}^{\infty} x_n e_n) = x_m$ are in E_* and $\hat{E}$ has order continuous norm, then E is a projection band of $\hat{E}$ whose orthogonal band is* $\operatorname{Ker} m_E$.

PROOF. Let $\hat{x} = (x_n)_{\mathscr{U}} \in \hat{E}$ be such that $m_E(\hat{x}) = 0$. Let $u = m_E(|x|)$. For each $i \in I$, $u(i) = \langle u, e_i^*\rangle = \lim_{\mathscr{U}} \langle |x_n|, e_i^*\rangle = \lim_{\mathscr{U}} |x_n(i)| = |\lim_{\mathscr{U}} x_n(i)| = 0$. Therefore $u = 0$ and $\operatorname{Ker} m_E$ is a sublattice of $\hat{E}$. In fact, since m_E is positive, $\operatorname{Ker} m_E$ is an ideal of $\hat{E}$. If $\hat{E}$ has order continuous norm, $\operatorname{Ker} m_E$ is a projection band of $\hat{E}$ ([19], theorem II-5.14(d)). We prove that if E is a sequence Banach lattice satisfying the hypothesis of the proposition, the orthogonal band to $\operatorname{Ker} m_E$ is E. We claim that if $0 \leqq |\hat{z}| \leqq u$, $\hat{z} \in \operatorname{Ker} m_E$ and $u \in E$, then $\hat{z} = 0$. It is sufficient to consider the case $\hat{z} \geqq 0$. Take $\hat{z} = (z_n)_{\mathscr{U}}$ with $0 \leqq z_n \leqq u$. Let (e_n) be the Schauder basis of E and let S_m and Q_m be the projections defined above. Then, given $\varepsilon > 0$, there exists $q \in \mathbf{N}$ such that $\| Q_q u \| < \varepsilon/2$. For $i \in (1, 2, \ldots, q)$, $\lim_{\mathscr{U}} z_n(i) = \langle m_E(\hat{z}), e_i^*\rangle = 0$. Let $M = \sup(\| e_i \|: i = 1, \ldots, q)$. Let $U_i \in \mathscr{U}$ be such that $|z_n(i)| < (2qm)^{-1}\varepsilon$ for each $n \in U_i$. Let $U := \bigcap_{i=1}^{q} U_i$. Let $n \in U$:

$$\| z_n \| \leqq \| S_q z_n \| + \| Q_q z_n \| < q(2qM)^{-1}\varepsilon \cdot M + \varepsilon/2.$$

For each $n \in U \in \mathscr{U}$, $\| z_n \| < \varepsilon$. Hence, $\hat{z} = 0$. Let $\hat{z} \in \hat{E}$, $u \in E$ be such that $0 \leqq \hat{z} \leqq u$. Then, $0 \leqq |\hat{z} - m_E(\hat{z})| \leqq 2u$, $\hat{z} - m_E(\hat{z}) \in \operatorname{Ker} m_E$. In consequence $\hat{z} - m_E(\hat{z}) = 0$. E is a closed ideal of $\hat{E}$. Now it is straightforward to prove that E and $\operatorname{Ker} m_E$ are complementary bands and $\hat{E} = E + \operatorname{Ker} m_E$.

REMARK 2.3. If E satisfies a lower p-estimate ([14], def. 1.f.4) then $\hat{E}$ satisfies a lower p-estimate. Therefore, $\hat{E}$ has order continuous norm.

We introduce here some notation we will need also is Section 3. Recall that $T \in \mathscr{L}(E)$ is called an "upper (lower) semi-Fredholm" operator if $T(E)$ is closed and its kernel (cokernel) Ker $T(E/T(E))$ is finite dimensional. The set of all upper (lower) semi-Fredholm operators will be denoted by $\Phi_+(E)$ (Φ (E), respectively). $\Phi(E) = \Phi_+(E) \cap \Phi_-(E)$ is the set of the so-called Fredholm operators. The quotient algebra $\mathscr{L}(E)/\mathscr{LC}(E)$, where $\mathscr{LC}(E)$ is the set of compact operators on E, will be denoted by $C(E)$ and π will be the canonical projection from $\mathscr{L}(E)$ onto $C(E)$.

LEMMA 2.4. *Let E be a dual Banach space and let E_* be a predual of E. Let $E_0 := \operatorname{Ker} m_E$, where m_E is the projection defined in Proposition 2.1. Then, the map L from $C(E_*)$ into $\mathscr{L}(E_0)$ given by $L(\pi(S)) := \hat{S}' \mid E_0$ is a continuous linear map such that $L(\pi(S)\pi(S_1)) = L(\pi(S_1))L(\pi(S))$, $S, S_1 \in \mathscr{L}(E_*)$.*

PROOF. Since $\hat{K}' \mid E_0 = 0$ when $K \in \mathscr{LC}(E_*)$, the map L is a well defined linear homomorphism from $C(E_*)$ into $\mathscr{L}(E_0)$. The other assertions are easy consequences of the definition of L and the inequality $\|L(\pi(S))\| \leqq \|\pi(S)\|$, $S \in \mathscr{L}(E_*)$.

THEOREM 2.5. *Let E be a dual Banach space. Let $T \in \mathscr{L}(E)$ be a dual operator. Then, $T \in \Phi(E)$ iff $t_0 := \hat{T} \mid E_0$ is bijective.*

PROOF. Let E_* be a predual of E. Let $S \in \mathscr{L}(E_*)$ be such that $S' = T$. If T is a Fredholm operator, also S is. Thus, there exists $S_1 \in \mathscr{L}(E_*)$ such that $\pi(S_1)\pi(S) = \pi(S)\pi(S_1) = \pi(I)$. Then, $L(\pi(S)\pi(S_1)) = L(\pi(S_1))L(\pi(S)) = I$. But $L(\pi(S)) = T_0$. To prove the converse, assume that T_0 is a bijection of E_0. If Ker T is infinite dimensional, then repeated applications of the Riesz lemma yield a sequence (x_n) in Ker T such that $\|x_n\| = 1$ and $\|x_n - x_m\| \geqq \frac{1}{2}$, $n \neq m$. Let $\hat{x} = (x_n)_{\mathscr{U}} \in \hat{E}$. Since T_0 is injective, $m_E(\hat{x})$ is non-zero. Let $\hat{z} = \hat{x} - m_E(\hat{x})$. Since (x_n) has no convergent subsequence, $\hat{z} \neq 0$. Thus, $\hat{z} \in E_0$, $\hat{z} \neq 0$ and $T_0\hat{z} = 0$, contrary to our assumptions on T_0. Therefore, Ker T is finite dimensional. Assume that $T(E)$ is not closed. Let P be a projection on E such that Ker P = Ker T. Then, $E = \operatorname{Ker} P \oplus P(E)$ and the restriction of T to $P(E)$ is injective. Since the range of $T \mid P(E)$ is not closed, there exists a sequence (x_n) in $P(E)$ such that $\|x_n\| = 1$ and $Tx_n \to 0$ as $n \to \infty$. Let $\hat{x} = (x_n)_{\mathscr{U}}$. Since T_0 is injective, $m_E(\hat{x}) \neq 0$. Let $\hat{z} = \hat{x} - m_E(\hat{x})$. Since $x_n \in P(E)$ and T is injective on $P(E)$, $\hat{z}$ cannot be zero. But $m_E(\hat{z}) = 0$ and $T_0\hat{z} = 0$. This contradiction proves that $T(E)$ is closed in E. Therefore, $T \in \Phi_+(E)$. Now, we prove that $\dim(E/T(E))$ is finite. Otherwise, by repeated application of the Riesz lemma, there exists a sequence (x_n) in E satisfying:

$$\|x_n\| \leqq 2 \quad \text{and} \quad \operatorname{dist}(x_{n+1}, T(E) + \operatorname{span}(x_j : j = 1, \ldots, n)) \geqq 1.$$

Let $\hat{x} = (x_n)_{\mathcal{U}}$ and let $\hat{z} = \hat{x} - m_E(\hat{x})$. Since (x_n) has no convergent subsequence, $\hat{z} \neq 0$. Consider the equation $T_0\hat{y} = \hat{z}$. This equation cannot have a solution $\hat{y} \in E_0$. Otherwise, there exist subsequences (y_{n_j}) and (x_{n_j}) such that $\lim_{j\to\infty} \| Ty_{n_j} - x_{n_j} + m_E(\hat{x})\| = 0$ and one can find integers k, j $(k > j)$ such that:

$$\| T(y_{n_j} - y_{n_k}) - x_{n_j} + x_{n_k} \| < \tfrac{1}{2}$$

in contradiction to our election of (x_n).

The argument in the proof of Theorem 2.5 is based on theorems 1 and 2 in [2]. Recalling that $\sigma_{\text{ess}}(T) := (z \in \mathbf{C}\colon z - T \notin \Phi(E))$ we state the following corollaries:

Corollary 2.6. *Let T be a dual operator on a dual Banach space E. Then, $\sigma_{\text{ess}}(T) = \sigma(\hat{T} \mid E_0)$.*

Corollary 2.7. *Let E be a dual Banach lattice satisfying the assumptions in Proposition 2.2. Let $0 \leqq T \in \mathcal{L}(E)$ be a dual operator. Then,*

$$r_{\text{ess}}(T) := \sup(|z| : z \in \sigma_{\text{ess}}(T)) \in \sigma_{\text{ess}}(T).$$

If one of the following conditions
(a) $((\lambda - r_{\text{ess}}(T))\pi(R(\lambda, T))\colon \lambda > r_{\text{ess}}(T))$ *is bounded*
(b) *T is a lattice homomorphism*
holds, then $(z \in \sigma_{\text{ess}}(T)\colon |z| = r_{\text{ess}}(T))$ is cyclic.

Proof. Let $T_0 := \hat{T} \mid E_0$. By Proposition 2.2, T_0 is a positive operator on the Banach lattice E_0. Hence, $r(T_0) \in \sigma(T_0)$ ([19], prop. V-4.1). By Corollary 2.6, $r_{\text{ess}}(T) \in \sigma_{\text{ess}}(T)$. If T satisfies (a) or (b), T_0 satisfies
(a′) $((\lambda - r(T_0))R(\lambda, T_0)\colon \lambda > r(T_0))$ *is bounded*
(b′) *T_0 is a lattice homomorphism*
respectively. In each one of these cases ([19], theorems V-4.4 and V-4.9) the set $(z \in \sigma(T_0)\colon |z| = r(T_0))$ is cyclic. By Corollary 2.6 the set $(z \in \sigma_{\text{ess}}(T)\colon |z| = r_{\text{ess}}(T))$ is cyclic.

Before stating the next corollary let us recall that $T \in \mathcal{L}(E)$ is called a Riesz operator if $\sigma_{\text{ess}}(T) = (0)$. Let $z \in \sigma(T)$. We say that z is a Riesz point of $\sigma(T)$ if z is a pole of $R(s, T) := (s - T)^{-1}$ such that the residuum is of finite rank. Then, T is a Riesz operator iff all the points in $\sigma(T)$, except 0, are Riesz points of $\sigma(T)$. This is a well known result, but it can be obtained from the results in the next section. Now, we state:

COROLLARY 2.8. *Let E be a Banach lattice as in Corollary 2.7. Let $S, T \in \mathscr{L}(E)$ be dual operators such that $0 \leqq S \leqq T$. If T is a Riesz operator, also S is.*

PROOF. Obviously, $0 \leqq S_0 \leqq T_0$ where $S_0 := \hat{S} \mid E_0$ and $T_0 := \hat{T} \mid E_0$. Then, $0 \leqq r(S_0) \leqq r(T_0)$. By Corollary 2.6, $r(T_0) = 0$. Again by Corollary 2.6, $r_{ess}(S) = r(S_0) = 0$.

3. A characterization of $\sigma_{ess}(T) \cap \partial_\infty \sigma(T)$

If T is a bounded operator on a Banach space E, we denote by $\sigma_W(T)$ the Weyl spectrum of T. We recall that

$$\sigma_W(T) = (z \in \mathbf{C}: z - T \notin \Phi_0(E)) = \cap (\sigma(T+K): K \in \mathscr{L}\mathscr{C}(E)),$$

where $\Phi_0(T)$ stands for the Fredholm operators of index zero. Finally, if T is a semi-Fredholm operator, the index of T is $i(T) := \dim \operatorname{Ker} T - \dim E/T(E)$.

LEMMA 3.1. *Let E be a Banach space and let $T \in \mathscr{L}(E)$. Let $\partial_\infty \sigma(T)$ be the set of points $z \in \sigma(T)$ such that all neighborhoods of z intersect the unbounded connected component of $\rho(T)$. If $\lambda \in \partial_\infty \sigma(T)$ is not an isolated point of $\sigma(T)$, then $\lambda - T \notin \Phi_+(E)$.*

PROOF. Since $\lambda \in \partial_\infty \sigma(T)$ and is not isolated in $\sigma(T)$, it is a consequence of Gohberg's theorem ([10], corol. 11.5) that $\lambda \in \sigma_W(T)$. If $\lambda - T \in \Phi_+(E)$, $(\lambda - T)(E)$ is closed and $\dim \operatorname{Ker}(\lambda - T)$ is finite. Using [12], lemma 4.5, there exists $t > 0$ such that if $|\lambda - \mu| < t$, then $(\mu - T) \in \Phi_+(E)$ and $i(\lambda - T) = i(\mu - T)$. But $(\mu \in \mathbf{C}: |\lambda - \mu| < t)$ intersects the resolvent set. Therefore, $i(\lambda - T) = i(\mu - T) = 0$ for all μ such that $|\lambda - \mu| < t$. Hence, $(\lambda - T) \in \Phi_0(E)$, in contradiction with $\lambda \in \sigma_W(T)$.

LEMMA 3.2. *Let E be a dual Banach space and let $T \in \mathscr{L}(E)$ be a dual operator. If $T \notin \Phi_+(E)$ then there exists $\hat{y} \in \hat{E}$, $\hat{y} \neq 0$, such that $m_E(\hat{y}) = 0$ and $\hat{T}\hat{y} = 0$.*

PROOF. If $T \notin \Phi_+(E)$, by corollary 4.12 in [12], there exists a normalized sequence (x_n) in E having no convergent subsequence such that $Tx_n \to 0$ as $n \to \infty$. Let $\hat{x} = (x_n)_{\mathscr{U}}$ and let $\hat{y} = \hat{x} - m_E(\hat{x})$. Then, $\hat{y} \in \hat{E}$, $\hat{y} \neq 0$, $m_E(\hat{y}) = 0$ and $\hat{T}\hat{y} = 0$.

PROPOSITION 3.3. *Let E be a Banach space and let $T \in \mathscr{L}(E)$. Let $\lambda \in \partial_\infty \sigma(T)$ be such that* $\dim \operatorname{Ker}(\lambda - \hat{T})$ *is finite. Then, λ is a pole of the resolvent $R(.,T)$ and its residuum is a projection of finite rank. Conversely, if $\lambda \in \partial_\infty \sigma(T)$ is a pole of $R(.,T)$ with a residuum of finite rank, then* $\dim \operatorname{Ker}(\lambda - \hat{T})$ *is finite.*

PROOF. If $\dim \operatorname{Ker}(\lambda - \hat{T})$ is finite, by proposition 3.2 in [7], we know that λ is an eigenvalue and $\dim \operatorname{Ker}(\lambda - T) = \dim \operatorname{Ker}(\lambda - \hat{T})$. Then, $\lambda \in \Phi_+(E)$. Otherwise, by corollary 4.12 in [12] there exists a normalized sequence (x_n) in E having no convergent subsequence such that $(\lambda - T)x_n$ converges to zero as $n \to \infty$. Let $\hat{x} = (x_n)_{\mathcal{U}}$. Then, $\hat{T}\hat{x} = \lambda\hat{x}$ and $\hat{x}$ is linearly independent of $\operatorname{Ker}(\lambda - T)$. Otherwise, let $u_1, \ldots, u_p$ be a basis of $\operatorname{Ker}(\lambda - T)$ and let $a, a_1, \ldots, a_p \in \mathbf{C}$, not all zero, such that $a\hat{x} + a_1\hat{u}_1 + \cdots + a_p\hat{u}_p = 0$, where $\hat{u}_i$ is the canonical image of $u_i \in E$ in $\hat{E}$. Since $a \neq 0$,

$$\hat{x} = a^{-1}a_1\hat{u}_1 + a^{-1}a_2\hat{u}_2 + \cdots + a^{-1}a_p\hat{u}_p.$$

Hence, (x_n) has a convergent subsequence. This contradiction implies that $\dim \operatorname{Ker}(\lambda - \hat{T}) > \dim \operatorname{Ker}(\lambda - T)$. Therefore, $(\lambda - T) \in \Phi_+(E)$. By Lemma 3.1, λ is isolated in $\sigma(T)$. In these conditions it is well known that λ is a pole of $R(.,T)$ with a residuum of finite rank. The converse is a consequence of the convergence of the integral

$$(2\pi i)^{-1} \int_\Gamma R(z,T)dz$$

(where Γ is a circle of center λ and sufficiently small radius) in the norm topology of $\mathcal{L}(E)$ and the continuity of the map $T \to \hat{T}$ with respect to the norm topologies of $\mathcal{L}(E)$ and $\mathcal{L}(\hat{E})$.

THEOREM 3.4. *Let E be a Banach space and let $T \in \mathcal{L}(E)$. We denote by C_1, C_2, C_3, C_4 the following sets*:

$C_1 = \partial_\infty\sigma(T) \cap \sigma_W(T)$,

$C_2 = \partial_\infty\sigma(T) \cap \sigma_{ess}(T)$,

$C_3 = \partial_\infty\sigma(T) \cap (z \in \mathbf{C}\colon z - T \notin \Phi_+(E))$,

$C_4 = \partial_\infty\sigma(T) \cap (z \in \mathbf{C}\colon \dim \operatorname{Ker}(z - \hat{T})$ *is infinite*).

Then, $C_1 = C_2 = C_3 = C_4$. If E is a dual Banach space, $T \in \mathcal{L}(E)$ is a dual operator and $C_5 := \partial_\infty\sigma(T) \cap (z \in \mathbf{C}\colon \exists \hat{y} \in \hat{E},\ \hat{y} \neq 0,\ m_E(\hat{y}) = 0$ and $\hat{T}\hat{y} = z\hat{y})$, then $C_5 = C_4$.

PROOF. By Lemma 3.1, $C_1 \subseteq C_3$. Let $z \in C_3$. According to lemma 5.1 in [12], $z - T'' \in \Phi_+(E'')$. Let $K \in \mathcal{LC}(E')$. By Lemma 3.2, there exists $\hat{y} \in \hat{E}''$, $\hat{y} \neq 0$ such that $m_{E''}(\hat{y}) = 0$ and $\hat{T}''\hat{y} = z\hat{y}$. Since $\hat{K}'\hat{y} = 0$, $(\hat{T}'' + \hat{K}')\hat{y} = z\hat{y}$. Therefore, $z \in \sigma(T'' + K') = \sigma(T' + K)$. Since $K' \in \mathcal{LC}(E')$ when $K \in \mathcal{LC}(E)$, this implies that

$$z \in \bigcap(\sigma(T+K)\colon K \in \mathcal{LC}(E)) =: \sigma_W(T).$$

Then, $z \in C_1$. Since $\sigma_{ess}(T) \subseteq \sigma_W(T)$, $C_2 \subseteq C_1$. If $z \in C_1 = C_3$, $z - T \notin \Phi_+(E)$. Hence, $z - T \notin \Phi(E)$ and $z \in C_2$. If $z \notin C_2$, $z - T \in \Phi(E)$. By Lemma 3.1, z is isolated in $\sigma(T)$. Since $z - T \in \Phi_+(E)$, z is a pole of $R(.,T)$ with a residuum of finite rank. By Proposition 3.3, $z \notin C_4$. If $z \notin C_4$, by Proposition 3.3, z is a pole of $R(.,T)$ with a residuum of finite rank. Using lemma 11.2.3 in [9], $z - T \in \Phi(E)$, i.e. $z \notin C_2$. We have proved that $C_2 = C_4$. Let E be a dual Banach space and let $T \in \mathscr{L}(E)$ be a dual operator. Let $z \in C_4$. If $\dim \operatorname{Ker}(z - T)$ is finite, let $\hat{y} \in \operatorname{Ker}(z - \hat{T})$, $\hat{y} \notin \operatorname{Ker}(z - T)$. Define $\hat{z} = \hat{y} - m_E(\hat{y})$. Then, $\hat{z} \neq 0$, $\hat{z} \in \operatorname{Ker} m_E$ and $\hat{T}\hat{z} = z\hat{z}$. If $\dim \operatorname{Ker}(z - T)$ is infinite, let (x_n) be a normalized sequence in $\operatorname{Ker}(z - T)$ having no convergent subsequence. Let $\hat{x} = (x_n)_{\mathscr{U}}$ and let $\hat{z} = \hat{x} - m_E(\hat{x})$. Again, $\hat{z} \in \hat{E}$, $\hat{z} \neq 0$, $m_E(\hat{z}) = 0$ and $\hat{T}\hat{z} = z\hat{z}$. Finally, let $z \in C_5$. If $z \notin C_4$, $\dim \operatorname{Ker}(z - \hat{T})$ is finite. By Proposition 3.3, $\operatorname{Ker}(z - \hat{T}) = \operatorname{Ker}(z - T)$. Hence, $z \notin C_5$. The theorem is proved.

4. On the peripheral spectrum of positive operators

The main purpose of this section is to prove the following result:

THEOREM 4.1. *Let E be a Banach lattice. Let $S, T \in \mathscr{L}(E)$ be such that $0 \leqq S \leqq T$ and $r(S) = r(T)$. If $r(T)$ is a Riesz point of $\sigma(T)$, then $r(S)$ is a Riesz point of $\sigma(S)$.*

We divide the proof in a series of propositions which are also interesting by themselves.

The next proposition was also used in a different context in [15], but we include here its proof for the sake of completeness.

PROPOSITION 4.2. *Let E be a Banach lattice. Let $S, T \in \mathscr{L}(E)$ be such that $0 \leqq S \leqq T$ and $r(T) = 1$. Assume that T is irreducible and 1 is a Riesz point of $\sigma(T)$. If 1 is an eigenvalue of S', then $S = T$.*

PROOF. Since 1 is a Riesz point of $\sigma(T)$, there exists $u \in E_+$ such that $Tu = u$ ([18], Appendix). Since T is irreducible, u is a quasi-interior point of E_+. Since 1 is an eigenvalue of S', there exists $f \in E'$ such that $S'f = f$. Then, $|f| \leqq S'|f| \leqq T'|f|$ and $\langle T'|f| - |f|, u\rangle = \langle |f|, Tu - u\rangle = 0$. Since u is a quasi-interior point of E_+, $T'|f| = S'|f| = |f|$. It follows from the irreducibility of T that $|f|$ is a strictly positive linear form on E. Let $0 \leqq x \in E$. Then, $0 \leqq Sx \leqq Tx$ and $\langle Tx - Sx, |f|\rangle = 0$. Since $|f|$ is strictly positive $Tx = Sx$. Therefore, $T = S$.

THEOREM 4.3. *Let E be a Banach lattice. Let $S, T \in \mathscr{L}(E)$ be such that*

$0 \leqq S \leqq T$ and $r(T)=1$. If T is irreducible and 1 is a Riesz point of $\sigma(T)$, then either $r(S)<r(T)$ or $S=T$.

PROOF. Assume that $r(S)=r(T)$ and 1 is not an eigenvalue of S'. Using proposition V-1.4 in [19], 1 is in the point spectrum of S'. We observe that there exists $\hat{y} \in \hat{E}'$, $\hat{y} \geqq 0$ and non-zero such that $\hat{S}'\hat{y}=\hat{y}$. Indeed, since $R(z,S')$ is not bounded when $z \to 1+$, there exist $0 \leqq x' \in E'$, $x' \neq 0$ and a sequence $z_n \to 1+$ such that $R(z_n,S')x'$ is unbounded when $n \to \infty$. Let $v_n := \| R(z_n,S')x' \|^{-1} R(z_n,S')x'$. Let $\hat{y}=(v_n)_{\mathscr{U}}$. Then, $0 \leqq \hat{y} \in E'$, $\|\hat{y}\|=1$, $S'v_n = z_n v_n - \| R(z_n,S')x' \|^{-1} x'$. Hence, $\hat{S}'\hat{y}=\hat{y}$. Let $m_{E'}$ be the projection defined as in Proposition 2.1. Then, $S'm_{E'}(\hat{y})=m_{E'}(\hat{y})$. Since 1 is not an eigenvalue of S', $m_{E'}(\hat{y})=0$. Let P be the residuum of $R(z,T)$ in $z=1$. Since T is irreducible, 1 is a first order pole of $R(z,T)$ and $P=f \otimes u$ where f is a strictly positive linear form on E and u is a quasi-interior point of E_+ ([19], corol. to prop. V-5.1 and theorem V-5.2). Then, the residuum of $R(z,\hat{T}')$ in $z=1$ is $\hat{P}'=\hat{u} \otimes \hat{f}$ where u represents the linear form on E' given by $\hat{u}((g_n)_{\mathscr{U}})=\lim_{\mathscr{U}}\langle u, g_n\rangle$. Let $R:=(\hat{g} \in \hat{E}' : \hat{P}'|g|=0)$, R is a T'-invariant ideal of E' and $r(\hat{T}' \mid R)<1$. Hence $r(\hat{S}' \mid R)<1$. From $\hat{S}'\hat{y}=\hat{y}$ and the last remark we conclude that $\hat{y} \notin R$. Then, $\hat{P}'\hat{y}=\hat{P}'|y| \neq 0$ and $\lim_{\mathscr{U}}\langle u, v_n\rangle = \langle \hat{u}, \hat{y}\rangle > 0$, in contradiction to $m_{E'}(\hat{y})=0$. Therefore, 1 is an eigenvalue of S'. By Proposition 4.2, $S=T$.

Fundamental for the proof of Theorem 4.1 is the next lemma, which is a refinement of lemma V-5.3 in [19]. We include a complete proof of it for the sake of completeness.

LEMMA 4.4. *Let E be a Banach lattice, let $T \in \mathscr{L}(E)$ and let J denote a closed T-invariant ideal of E; by $T_1 := T \mid J$ and $T_2 := T_J$ we denote the induced operators on J and E/J, respectively. If z_0 is a Riesz point of $\sigma(T_1)$ and z_0 is a Riesz point of $\sigma(T_2)$, then z_0 is a Riesz point of $\sigma(T)$.*

PROOF. By (II-5.5) cor. 1 in [19], the polar J^0 is a band in the order complete dual E' and hence, by theorem II-2.10 in [19], $E'=J^0+(J^0)^\perp$. With respect to this direct sum, the adjoint T' has a matrix representation

$$T' = \begin{pmatrix} T_2' & S \\ 0 & T_1' \end{pmatrix}$$

where J^0 and $(J^0)^\perp$ are identified with the duals of E/J and J, respectively, and where $S \in \mathscr{L}((J^0)^\perp, J^0)$. Hence, in a neighborhood of z_0, the resolvent of T' has the matrix representation

$$R(z, T') = \begin{pmatrix} R(z, T_2') & R(z, T_2')SR(z, T_1') \\ 0 & R(z, T_1') \end{pmatrix}$$

which shows that z_0 is a pole of $R(z, T')$. Let us call by P_1 and P_2 the residuums of $R(z, T_1)$ and $R(z, T_2)$, respectively, in $z = z_0$. Let P be the residuum of $R(z, T')$ in $z = z_0$. then,

$$P = \begin{pmatrix} P_2' & A \\ 0 & P_1' \end{pmatrix}$$

where $A \in \mathscr{L}((J^0)^\perp, J^0)$. Since P is a projection, $P^2 = P$, and a simple calculus gives $A = P_2'A + AP_1'$. Since P_1' and P_2' are projections of finite rank, A is an operator of finite rank. Therefore, z_0 is a Riesz point of $\sigma(T')$. Hence, z_0 is a Riesz point of $\sigma(T)$.

The proof of Theorem 4.1 is based on the proof of theorem V-5.5 in [19].

Proof of Theorem 4.1. We may assume without loss of generality that $r(T) = 1$. The proof is divided into three steps.

(1) First, assume that $z = 1$ is a first order pole of $R(z, T)$ and its residuum P is strictly positive, i.e. $(x \in E\colon P|x| = 0) = 0$. It follows from [19], prop. III-11.5 that $P(E)$ is a finite dimensional sublattice of E and can be identified with $\mathbf{C}^n$ for some $n \in \mathbf{N}$ ([19], II-3.9 cor. 1). If $(e_1, e_2, \dots, e_n)$ denotes the standard basis of $\mathbf{C}^n$ and if J_k denotes the closed ideal of E generated by e_k, $k = 1, \dots, n$, we conclude from proposition III-8.5 in [19] that each $T_k := T \mid J_k$ is irreducible. T_k is an irreducible operator and $r(T_k) = 1$ is a Riesz point of $\sigma(T_k)$. Since $0 \leqq S_k \leqq T_k$, by Theorem 4.3, either $r(S_k) < r(T_k)$ or $S_k = T_k$ and 1 is a Riesz point of $\sigma(S_k)$. If $J := \sum_k J_k$, J is a closed, T-invariant ideal of E, $P(E) \subseteq J$ and, if T_j denotes the induced operator on E/J, $r(T_J) < 1$. Therefore, $r(S_J) < 1$. On the other hand, $r(S \mid J) = r(T \mid J) = 1$ and 1 is a Riesz point of $\sigma(S \mid J)$. Using Lemma 4.4, we conclude that 1 is a Riesz point of $\sigma(S)$.

(2) Next, if $z = 1$ is a first order pole of $R(z, T)$ but its residuum P is not strictly positive, the operators S_R and T_R induced on E/R by S and T, respectively, where $R = (x \in E\colon P|x| = 0)$, satisfy $0 \leqq S_R \leqq T_R$ and $r(T_R) = 1$ is a Riesz point of $\sigma(T_R)$ with residuum P_R strictly positive. We are under the conditions of step (1) hence, since $r(S_R) = 1$, 1 is a Riesz point of $\sigma(S_R)$. On the other hand, $0 \leqq r(S \mid R) \leqq r(T \mid R) < 1$. An application of Lemma 4.4 proves that $r(S) = 1$ is a Riesz point of $\sigma(S)$.

(3) Finally, assume that $z = 1$ is a pole of $R(z, T)$ of order $k > 1$. Consider the operator $Q = \lim_{s \to 1} (s - 1)^k R(s, T)$. We have $Q^2 = 0$. Let $J =$

$(x \in E\colon Q|x|=0)$. Since Q is positive and commutes with T, J is a closed T-invariant ideal of E such that $R(z, T \mid H) = R(z, T)|J$ has at $z = 1$ a pole of order $< k$ while on E/J, $R(z, T_J)$ has at $z = 1$ a pole of order 1, where T_J is the operator induced by T on E/J. Then $0 \leqq S_J \leqq T_J$ and $r(T_J) = 1$ is a Riesz point of $\sigma(T_J)$ and a simple pole of $R(z, T_J)$. Using step (2), either $r(S_J) < 1$ or 1 is a Riesz point of $\sigma(S_J)$. By induction on k we reduce the proof to step (2) and an application of Lemma 4.4 finishes the proof.

Corollary 4.5. *Let E be a Banach lattice. Let $(S(t)\colon t \geqq 0)$ $((T(t)\colon t \geqq 0))$ be a strongly continuous semigroup with generator $(A, D(A))$ $((B, D(B)))$. Assume that $0 \leqq S(t) \leqq T(t)$ for all $t \geqq 0$ and $s(A) := \sup(\mathrm{Re}\, z\colon z \in \sigma(A)) = \sup(\mathrm{Re}\, z\colon z \in \sigma(B)) =: s(B)$. If $s(B)$ is a Riesz point of $\sigma(B)$, then $s(A)$ is a Riesz point of $\sigma(A)$.*

Proof. We may assume without loss of generality that $s(A) = s(B) = 0$. Let us fix $\lambda > 0$. Then, $0 \leqq R(\lambda, A) \leqq R(\lambda, B)$, $r(R(\lambda, A)) = r(R(\lambda, B)) = \lambda^{-1}$ and λ^{-1} is a Riesz point of $\sigma(R(\lambda, B))$. By Theorem 4.1, λ^{-1} is a Riesz point of $\sigma(R(\lambda, A))$. Therefore, $s(A) = 0$ is a Riesz point of $\sigma(A)$.

Corollary 4.6. *Let E be a Banach lattice. Let $S, T \in \mathscr{L}(E)$ be such that $0 \leqq S \leqq T$ and $r(T) = 1$. Suppose that T is uniformly ergodic with finite-dimensional fixed space $(x \in E\colon Tx = x)$. Then, S is uniformly ergodic with finite-dimensional fixed space.*

Proof. As a consequence of some results of Karlin ([11]) and Wolff ([21]) (a) T (or S) is uniformly ergodic with finite-dimensional fixed space if and only if (b) $r(T) \leqq 1$ (resp. $r(S) \leqq 1$), $z = 1$ is a pole of $R(z, T)$ (resp. $R(z, S)$) of order, at most, one and the fixed space of $T(S)$ is finite-dimensional. Then, Corollary 4.6 is a consequence of Theorem 4.1 if we observe that if $0 \leqq S \leqq T$ and $z = 1$ is a first order pole of $R(z, T)$ and a pole of $R(z, S)$, then $z = 1$ is a pole of $R(z, S)$ of order one.

Remark 4.7. Observe that $0 \leqq S \leqq T$, $r(S) = r(T)$ and $z = 1$ is a pole of $R(z, T)$ does not imply that $z = 1$ is a pole of $R(z, S)$. Take, for example, $E = l^2$, $T = I$ and S the operator on l^2 given by the diagonal matrix with entries $(1, 1 - 2^{-1}, 1 - 3^{-1}, \ldots, 1 - n^{-1}, \ldots)$.

5. On compact positive perturbations of positive operators

We prove in this section some results on compact positive perturbations of certain classes of positive operators. Theorem 5.1 below is the type of result we are looking for in this section. The peripheral spectrum of an operator $T \in \mathscr{L}(E)$, i.e. the set $(z \in \sigma(T): |z| = r(T))$ will be denoted by $\pi\sigma(T)$.

THEOREM 5.1. *Let E be a Banach lattice. Let $0 \leqq T \in \mathscr{L}(E)$ be an irreducible operator and let $0 \leqq K \in \mathscr{L}(E)$ be compact and non-null. Suppose that there exists a strictly positive linear form f on E such that $(T' + K')f \leqq r(T' + K')f$. Then, one of the two following assertions is true:*

(a) $r(T+K) > r(T)$,

(b) $\pi\sigma(T+K) = \pi\sigma(T)$ *and T has no eigenvalues in* $\pi\sigma(T)$.

PROOF. If $r(T) < r(T+K)$ we are done. Otherwise, we may assume without loss of generality that $r(T+K) = r(T) = 1$. If $a \in \pi\sigma(T)$ is an eigenvalue of T, let $x \in E$, $x \neq 0$, be such that $Tx = ax$. Then, $|x| \leqq T|x|$ and

$$0 \leqq \langle T|x| - |x|, f\rangle \leqq \langle |x|, T'f - f\rangle \leqq 0.$$

Since f is strictly positive, $T|x| = |x|$. By the irreducibility of T, $|x|$ is a quasi-interior point of E_+. Then, $T'f = f$. Hence $K'f = 0$, i.e., $\langle Kx, f\rangle = 0$ for all $0 \leqq x \in E$. Since f is strictly positive $K = 0$. This contradiction proves that T has no eigenvalues in $\pi\sigma(T)$. Then $\pi\sigma(T) \cap \sigma_{ess}(T) = \pi\sigma(T)$ (it can be proved, for example, using the results of Section 3). As a consequence of Gohberg's theorem ([10], corol. 11.5),

$$\pi\sigma(T+K) = \pi\sigma(T) \cup (z \in \mathbf{C}: |z| = 1 \text{ and } z \text{ is a Riesz point of } \sigma(T+K)).$$

Let $a \in \pi\sigma(T+K)$ be a Riesz point of $\sigma(T+K)$. There exists $x \in E$, $x \neq 0$, such that $(T+K)x = ax$. By proposition V-5.1 in [19], there exists a surjective isometry $V: E \to E$ such that $a(T+K) = V^{-1}(T+K)V$. Since $T+K$ is positive, $\bar{a}$ is also a Riesz point of $\sigma(T+K)$. Then, $a\bar{a} = 1$ is also a Riesz point of $\sigma(T+K)$. On the other hand, we know that $1 \in \sigma_{ess}(T)$. Therefore, $1 \in \sigma_{ess}(T+K)$. From this contradiction, we conclude that $\pi\sigma(T+K) = \pi\sigma(T)$. The theorem is proved

COROLLARY 5.2. *Let $E = C(X)$ where X is a compact, Hausdorff topological space. Let $0 \leqq T \in \mathscr{L}(E)$ be irreducible and $0 \leqq K \in \mathscr{L}(E)$ be compact and non-null. Then, one of the two following assertions is true:*

(a) $r(T+K) > r(T)$,

(b) $\pi\sigma(T+K) = \pi\sigma(T)$ *and T has no eigenvalues on* $\pi\sigma(T)$.

PROOF. Since $E = C(X)$, there exists $0 \leqq f \in E'$ such that $(T' + K')f = r(T+K)f$ ([18], app. corol. 2.6). From the irreducibility of T it follows that f is strictly positive. Now, the corollary is a consequence of Theorem 5.1.

REMARK 5.3. A similar result to our Theorem 5.1 was proved in [15]. In fact, if we don't assume in Theorem 5.1 the existence of a strictly positive linear form f such that $(T'+K')f \leqq r(T+K)f$ but we suppose that there exists a positive eigenvector associated to $r(T)$, then the following are equivalent: (a) $r(T+K) > r(T)$, (b) $\pi\sigma(T+K)$ consists of poles of $R(z, T+K)$, (c) $T'+K'$ has a positive eigenvector associated to $r(T+K)$ ([15], Thm. 5.6). From this result, Moustakas deduces also his version of Corollary 5.2 ([15], corol. 5.7)

THEOREM 5.4. *Let E be a Banach lattice. Let $0 \leqq T \in \mathscr{L}(E)$ be a lattice homomorphism and let $0 \leqq K \in \mathscr{L}(E)$ be a compact operator such that $K'g \in E'$ is strictly positive when $g \in E'$ is strictly positive. Suppose that there exists $f \in E'$ strictly positive such that $(T'+K')f \leqq r(T+K)f$. Then, one of the two following assertions holds*:

(a) $r(T+K) > r(T)$,

(b) $\pi\sigma(T+K) = \pi\sigma(T) \cup (z \in \pi\sigma(T+K)$: *$z$ is a Riesz point of* $\sigma(T+K))$ *and T has no eigenvalues on* $\pi\sigma(T)$.

If $T+K$ is irreducible, the set of Riesz points of $\sigma(T+K)$ that appear in case (b) *is empty.*

PROOF. Assume without loss of generality that $r(T+K) = r(T) = 1$. If $a \in \pi\sigma(T)$ is an eigenvalue of T, let $x \in E$, $x \neq 0$, be such that $Tx = ax$. Since T is a lattice homomorphism, $T|x| = |x|$. Then,

$$0 \leqq \langle |x|, K'f \rangle \leqq \langle |x|, f - T'f \rangle \leqq \langle T|x| - |x|, f \rangle \leqq 0.$$

Hence, $K'f$ is not strictly positive, contradicting our assumptions on K. Then, T has no eigenvalues on $\pi\sigma(T)$, $\pi\sigma(T) \cap \sigma_{\mathrm{ess}}(T) = \pi\sigma(T)$ and as a consequence of Gohberg's theorem ([10], cor. 11.5),

$$\pi\sigma(T+K) = \pi\sigma(T) \cup (z \in \pi\sigma(T+K)\text{: } z \text{ is a Riesz point of } \sigma(T+K)).$$

If $T+K$ is irreducible, a similar argument to the one used in the proof of Theorem 5.1 proves that $\pi\sigma(T+K) = \pi\sigma(T)$.

COROLLARY 5.5. *Let $E = C(X)$ where X is a compact, Hausdorff topological space. Let $T \in \mathscr{L}(E)$ be a lattice homomorphism and $0 \leqq K \in \mathscr{L}(E)$ be compact and non-zero. Suppose that $T+K$ is irreducible and K is such that $K'g \in E'$ is strictly positive when $g \in E'$ is strictly positive. Then, one of the two following assertions holds*:

(a) $r(T+K) > r(T)$,

(b) $\pi\sigma(T+K) = \pi\sigma(T)$ *and T has no eigenvalues on* $\pi\sigma(T)$.

PROOF. It follows from Theorem 5.4 in the same way that Corollary 5.2 follows from Theorem 5.1.

There exist similar results for the generator of a semigroup of positive operators. We finish this section by proving them.

Before stating Theorem 5.6, let us recall some definitions and notations. A strongly continuous semigroup of positive operators $(S(t)\colon t \geqq 0)$ acting on the Banach lattice E is called irreducible if the only ideals of E that are invariant under the action of all operators of $(S(t)\colon t \geqq 0)$ are (0) and E. If $(S(t)\colon t \geqq 0)$ is a strongly continuous semigroup of operators on the Banach space E with generator $(A, D(A))$, we denote by $s(A)$ the number $\sup(\operatorname{Re} z\colon z \in \sigma(A))$. In what follows we assume that $s(A) > -\infty$. Then, $\sigma_+(A)$ will be the set $(z \in \sigma(A)\colon \operatorname{Re} z = s(A))$.

THEOREM 5.6. *Let E be a Banach lattice. Let $(S(t)\colon t \geqq 0)$ be a strongly continuous, irreducible semigroup of positive operators with generator $(A, D(A))$. Let $(T(t)\colon t \geqq 0)$ be another strongly continuous semigroup of operators on E such that $0 \leqq S(t) \leqq T(t)$ for all $t \geqq 0$ with generator $(A+K, D(A+K))$ where $K \in \mathcal{L}(E)$ is non-zero and $KR(z, A)$ is compact for some $z \in \rho(A)$ (hence, for all $z \in \rho(A)$). Suppose that there exists $f \in E'$ strictly positive such that*

$$(z_0 - s(A+K))R(z_0, A+K)'f \leqq f \qquad \text{for some } z_0 > s(A+K).$$

Then, one of the two following assertions holds:

(a) $s(A+K) > s(A)$,

(b) $\sigma_+(A+K) = \sigma_+(A)$ *and A has no eigenvalues on $\sigma_+(A)$.*

PROOF. Since $0 \leqq S(t) \leqq T(t)$ for all $t \geqq 0$ and

$$R(z, A+K)x = \int_0^{+\infty} e^{-zt} T(t)x\,dt \qquad \text{for all } z \text{ such that } \operatorname{Re} z > s(A+K)$$

([4], Lemma 3.2), if z is such that $\operatorname{Re} z > s(A+K)$, $z \in \rho(A)$, $0 \leqq R(z, A) \leqq R(z, A+K)$ and, in consequence, $s(A) \leqq s(A+K)$. If $s(A) < s(A+K)$ we are done. Otherwise, $s(A) = s(A+K)$. We may assume without loss of generality that $s(A+K) = 0$. If $ia \in \sigma_+(A)$, $a \in \mathbf{R}$, is an eigenvalue of A, let $x \in E$, $x \neq 0$, such that $Ax = iax$. Then, $(z - ia)R(z, A)x = x$ for all z such that $\operatorname{Re} z > 0$. Using the second resolvent formula, $zR(z + ia, A)x = x$, $\operatorname{Re} z > 0$. Hence, $|x| \leqq z_0 R(z_0, A)|x|$. Since $z_0 R(z_0, A)'f \leqq z_0 R(z_0, A+K)'f \leqq f$ we have

$$0 \leqq \langle z_0 R(z_0, A)|x| - |x|, f\rangle \leqq \langle |x|, z_0 R(z_0, A)'f - f\rangle \leqq 0.$$

Since f is strictly positive, $z_0R(z_0, A)|x| = |x|$. It follows from the irreducibility of $S(t)$ and the inequality

$$S(t)R(z_0, A)|x| = e^{z_0 t}\int_t^{+\infty} e^{-z_0 s}S(s)|x|\,ds \leqq e^{z_0 t}R(z_0, A)|x|$$

that $R(z_0, A)|x|$ is a quasi-interior point of E_+. Hence, $|x|$ is a quasi-interior point of E_+ and $z_0R(z_0, A)'f = f$. Using the formula

$$(*) \qquad R(z_0, A+K) = R(z_0, A) + R(z_0, A+K)KR(z_0, A)$$

it follows that

$$R(z_0, A)'K'R(z_0, A+K)'f = 0.$$

Hence,

$$\langle KR(z_0, A)u, R(z_0, A+K)'f\rangle = 0 \qquad \text{for all } u \in E_+.$$

Since $R(z_0, A+K)'f$ is strictly positive, $KR(z_0, A)u = 0$. If $u \neq 0$, $R(z_0, A)u$ is a quasi-interior point of E_+, hence, $K = 0$. From this contradiction we deduce that A has no eigenvalues in $\sigma_+(A)$. Let $ia \in \sigma_+(A)$. Then $(z_0 - ia)^{-1} \in \partial_\infty\sigma(R(z_0, A))$ and is not an eigenvalue of $R(z_0, A)$ ([6], prop. 1.1). Hence, $(z_0 - ia)^{-1} \in \sigma_{ess}(R(z_0, A))$. Using the formula (*) and the compactness of $KR(z_0, A)$, it follows that $(z_0 - ia)^{-1} \in \sigma_{ess}(R(z_0, A+K))$. Using Gohberg's theorem ([10], cor. 11.5) it follows that

$$\sigma_+(A+K) = \sigma_+(A) \cup (z \in \mathbf{C}\colon \operatorname{Re} z = 0 \text{ and } z \text{ is a pole of } R(.\,, A+K) \text{ with residuum of finite rank}).$$

Let ib be a pole of $R(z, A+K)$ with a residuum of finite rank. Let $u \in E$, $u \neq 0$ be such that $(z - ib)R(z, A+K)u = u$, $\operatorname{Re} z > 0$. Then, $zR(z+ib, A+K)u = u$, $\operatorname{Re} z > 0$, and $|u| \leqq wR(w, A+K)|u|$ for all $w > 0$. Since $z_0R(z_0, A+K)'f \leqq f$ and f is strictly positive, it follows that $z_0R(z_0, A+K)|u| \leqq |u|$. By ([6], cor. 1.7), there exists a surjective isometry $V\colon E \to E$ such that

$$R(z, A+K) = V^{-1}R(z+ib, A+K)V \qquad \text{for all } z \text{ such that } \operatorname{Re} z > 0.$$

Since ib is a pole of $R(z, A+K)$ with residuum of finite rank, by the similarity just proved, $z = 0$ is also a pole of $R(z, A+K)$ with a residuum of finite rank. Again, using the compactness of $KR(z, A)$ and formula (*), $z = 0$ is a pole of $R(z, A)$ with a residuum of finite rank. This is impossible since A has no eigenvalues on $\sigma_+(A)$. Therefore, $\sigma_+(A+K) = \sigma_+(A)$.

COROLLARY 5.7. *Let $E = C(X)$ where X is a compact, Hausdorff topological* *space. Let* $(S(t)\colon t \geqq 0)$, $(T(t)\colon t \geqq 0)$ *be strongly continuous semigroups of positive operators with generators* $(A, D(A))$ *and* $(A + K, D(A + K))$, *respectively, where* $K \neq 0$, $K \in \mathcal{L}(E)$ *is such that* $KR(z, A)$ *is compact for some* $z \in \rho(A)$. *Suppose that* $(S(t)\colon t \geqq 0)$ *is irreducible and* $0 \leqq S(t) \leqq T(t)$ *for all* $t \geqq 0$. *Then, one of the following assertions holds*:

(a) $s(A + K) > s(A)$,

(b) $\sigma_+(A + K) = \sigma_+(A)$ *and* A *has no eigenvalues in* $\sigma_+(A)$.

THEOREM 5.8. *Let* E *be a Banach lattice. Let* $(S(t)\colon t \geqq 0)$ *and* $(T(t)\colon t \geqq 0)$ *be strongly continuous semigroups of positive operators with generators* $(A, D(A))$ *and* $(A + K, D(A + K))$, *respectively, where* $K \in \mathcal{L}(E)$, $K \neq 0$, *is such that* $KR(z, A)$ *is compact for some* $z \in \rho(A)$. *Suppose that* $S(t)$ *are lattice homomorphisms for all* $t \geqq 0$ *and* $(T(t)\colon t \geqq 0)$ *is irreducible. Suppose that* $K'g \in E'$ *is strictly positive when* $g \in E'$ *is strictly positive and suppose*

(H) *there exists* $f \in E'$ *strictly positive such that*

$$(z_0 - s(A + K))R(z_0, A + K)'f \leqq f \qquad \text{for some } z_0 > s(A + K).$$

Then, one of the two following assertions holds:

(a) $s(A + K) > s(A)$,

(b) $\sigma_+(A + K) = \sigma_+(A)$ *and* A *has no eigenvalue on* $\sigma_+(A)$.

PROOF. If $s(A) = s(A + K)$, we may assume that $s(A + K) = 0$. If $ia \in \sigma_+(A)$, $a \in \mathbf{R}$ is an eigenvalue of A, let $x \in E$, $x \neq 0$, be such that $Ax = iax$. Then, $S(t) = e^{iat}x$ for all $t \geqq 0$. Since $S(t)$ is a lattice homomorphism for all $t \geqq 0$, $S(t)|x| = |x|$ for all $t \geqq 0$ and $z_0 R(z_0, A)|x| = |x|$. Using the formula (*) in the proof of Theorem 5.6 and hypothesis (H) it follows that $\langle K'R(z_0, A + K)'f, |x| \rangle = 0$. Since $(T(t)\colon t \geqq 0)$ is an irreducible semigroup, $g := R(z_0, A + K)'f$ is strictly positive and $\langle K'g, |x| \rangle = 0$, contradicting our hypothesis on K. A cannot have eigenvalues on $\sigma_+(A)$. A similar argument to the one used in the proof of Theorem 5.6 proves that $\sigma_+(A + K) = \sigma_+(A)$.

We end with the corresponding corollary when $E = C(X)$.

COROLLARY 5.9. *If* $E = C(X)$, *where* X *is a compact, Hausdorff topological space, the hypothesis* (H) *in Theorem* 5.8 *is unnecessary.*

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